SUPPLEMENTARY NOTES ON SECTION 2

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Although there are no explicit "exercises" in section 2 of `Applied_Math_Notes_Fall2020.pdf` per se, there are a number of concepts and theorems which inspired some discussion and exercises, to be found in this document. Contributions, suggestions, and corrections are most appreciated.

"Mathematics compares the most diverse phenomena and discovers the secret analogies that unite them."
- J. Fourier

- *Exploring systems with orthogonal rows and seeing if Gram-Schmidt helps*, **Theorem 2.2:** Earlier, we discussed converting a second-order linear differential equation to system of two, first-order differential equations (in lecture 3). That is,

$$y'' - 3y' + 2y = f \iff \begin{pmatrix} y' \\ y'' \end{pmatrix} = \begin{pmatrix} y' \\ 3y' - 2y + f \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix} + \begin{pmatrix} 0 \\ f \end{pmatrix}. \tag{1}$$

Theorem 2.2 (page 8) discusses a process called Gram-Schmidt which allows the creation of an orthonormal set of vectors in a pre-Hilbert space V from an arbitrary (non-empty) set of vectors in V. Let's re-visit the generalization of Equation (1) by considering the following

$$\Psi' = M\Psi + W \tag{2}$$

as a linear, inhomogeneous system of n ordinary differential equations, where M is a square matrix having real entries. Also consider an arbitrary set $\mathcal{R}_j^0 \equiv \{v_1, \cdots, v_n\}$ of n, vectors, with each $v_j \in \mathbb{R}^n$. One could then potentially use Gram-Schmidt 'orthonormalization' to find a set $\mathcal{R}_j$ of orthonormal vectors in $\mathbb{R}^n$ and we could even fill each j^{th} row of a matrix with the j^{th} entry of $\mathcal{R}_j$. What would be the effect on Equation (2) of such a matrix M having orthonormal rows?

Going back to Equation (1), if we generalize the coefficients ($3 \mapsto \alpha,\ 2 \mapsto \beta,\ \alpha, \beta \in \mathbb{R}$), we have

$$y'' - \alpha y' + \beta y = f \iff \begin{pmatrix} y' \\ y'' \end{pmatrix} = \begin{pmatrix} y' \\ \alpha y' - \beta y + f \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\beta & \alpha \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix} + \begin{pmatrix} 0 \\ f \end{pmatrix} \tag{3}$$

thus, orthogonal rows are only possible in this case if the inner product is identically zero for vectors $(0, 1)$ and $(-\beta, \alpha)$ in $\mathbb{R}^2$, which happens only when $\alpha = 0$ (assuming $\mathbb{R}^2$ has the standard, Euclidean inner product). If we require $\alpha = 0$, we have the following differential equation.

$$\left(\frac{d^2}{dx^2} + \beta\right) y = f. \tag{4}$$

Let's use the fact that, for Equation (4), the system and its solution are invariant to row permutations (on both sides of the equation); for instance, below is an equivalent system.

$$\begin{pmatrix} y'' \\ y' \end{pmatrix} = \begin{pmatrix} -\beta & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix} + \begin{pmatrix} f \\ 0 \end{pmatrix} \tag{5}$$

What value(s) can β take so that the matrix in Equation (5) has orthonormal rows? The β-restriction is of course enforced by the nature of the second row, which stems from the original differential equation we chose.

Let's go up to a 3×3 matrix and pick an arbitrary set of 3 vectors to start with and apply the Gram-Schmidt process. The original set of vectors will be

$$\mathcal{R}_j^0 \equiv \{v_1 \equiv (1,1,0), v_2 \equiv (1,2,0), v_3 \equiv (0,1,2)\}. \tag{6}$$

Assuming $\mathcal{R}_j^0 \subset \mathbb{R}^3$ with Euclidean inner product $(,)$ and the induced norm $\sqrt{(,)}$, the Gram-Schmidt 'orthonormalization' process is carried out below.

$$\begin{aligned} & v_1 \mapsto \frac{w_1}{\|w_1\|} \equiv \frac{1}{\sqrt{2}}(1,1,0), \\ & v_2 \mapsto v_2 - \frac{(v_2,w_1)}{(w_1,w_1)}w_1 = (1,2,0) - \frac{3}{2}(1,1,0) = (-1/2, 1/2, 0), \\ \text{and } & v_3 \mapsto v_3 - \frac{(v_3,w_1)}{(w_1,w_1)}w_1 - \frac{(v_3,w_2)}{(w_2,w_2)}w_2 = (0,1,2) - \frac{1}{2}(1,1,0) - (-1/2,1/2,0) = (0,0,2). \end{aligned} \tag{7}$$

If we use this new set of vectors as the rows of a matrix, we have $\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. Permuting the rows and creating a differential equation, we consider

$$\begin{pmatrix} y' \\ y'' \\ y''' \end{pmatrix} = \begin{pmatrix} -1/2 & 1/2 & 0 \\ 0 & 0 & 2 \\ 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix} \iff \begin{cases} y'' & = y'' \\ y' & = (-1/2)(y - y') \\ y''' & = (1/\sqrt{2})(y + y' + y'') \end{cases}. \tag{8}$$

It is not terribly clear how this is useful, but clearly if we had picked a system similar to Equation (5), like

$$\begin{pmatrix} y' \\ y'' \\ y''' \end{pmatrix} = \begin{pmatrix} & 1 & \\ & & 1 \\ -\beta & & \end{pmatrix} \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ f \end{pmatrix} \tag{9}$$

with driving function f, then we would again have $y''' = -\beta y + f \Rightarrow \left(\frac{d^3}{dx^3} + \beta\right) y = f$.

Lemma 1. *An n-dimensional linear system of the form*

$$\begin{pmatrix} y' \\ \vdots \\ y^{(k)} \end{pmatrix} = \begin{pmatrix} 0 & & & \\ \vdots & & \mathbb{I}_{k-1} & \\ 0 & & & \\ -\beta & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} y \\ \vdots \\ y^{(k-1)} \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ f \end{pmatrix} \tag{10}$$

with driving function f is equivalent to the k^{th} degree equation $\left(\frac{d^k}{dx^k} + \beta\right) y = f$.

Remark. Calling this a lemma may be overkill, but the intention is to reference it later. Notice the identity block's purpose is nothing but repeated self-identification.

Remark. Can one come up with a similar idea for Gram-Schmidt, i.e. can one start from an arbitrary set of n-dimensional vectors, orthonormalize the set through Gram-Schmidt, construct the coefficient matrix and then finally have a formula for the resulting single differential equation Υ? Suppose it is no longer a constant matrix, but a matrix of functions; would that change the expression for Υ?

- ***Exercise:*** (Linear algebra practice) Make a coefficient matrix and write a corresponding, non-homogeneous system of differential equations in matrix form (as in **Lemma 1**) for the k^{th} degree equation

$$\left[\left(\sum_{i=1}^{k} \frac{d^i}{dx^i}\right) + \beta\right] y = \left(\frac{d^k}{dx^k} + \frac{d^{k-1}}{dx^{k-1}} + \cdots + \frac{d}{dx} + \beta\right) y = f.$$

  (*Hint*: figure out where just one term from the sum ends up permuting a 1 in the matrix and pretend the rest of the terms in the sum vanish. Repeat.)

- Not so much a specific item in the notes, but Ling and I discussed for a while the generality of a pre-Hilbert space and how the dimension can be finite or infinite, and the consequences. For instance, Gram-Schmidt is exactly the same process with an inner product like $\int_X f\bar{g}\, d\mu$. One example of a finite-dimensional Hilbert space is $\mathbb{C}^n$.

- ***Exercise:*** prove that $\mathbb{C}^n$ is complete, $n \in \mathbb{N}^\times$ (hint: if you can prove it for one space, you can prove it for a Cartesian product of that one space).

- Throughout section 2, elements of dual spaces are mentioned here and there, so we discussed that as well. Let V be a vector space. Its dual, usually denoted V^* is a set of linear maps from $V \to \mathbb{K}$ (For our purposes, it seems that mostly $\mathbb{K} = \mathbb{R}, \mathbb{C}$). Notice how this definition also doesn't mention dimension at all; it applies whether V is finite- or infinite-dimensional. These maps are also called linear functionals.

  Example 1. Let V be a pre-Hilbert space with inner product $(,)$. If $w \in V^*$ and $v \in V$, where $w : v \in V \mapsto (w, v) \in \mathbb{K}$, then this is an example of a linear functional.

  Example 2. A definite integral is also a linear functional. We discussed how one can think of it "eating" functions within the box below. For example,

$$\int_\Omega \Box\, dx : f \in C^k(\mathbb{R}) \mapsto \int_\Omega f\ dx \in \mathbb{R}, \qquad \Omega \subset \mathbb{R}.$$

  This is redundant with **Example 1** for a particular choice of w and a particular inner product space. Can you come up with those choices?

  Example 3. Another example we started discussing is more of a bridge from finite to infinite-dimensional spaces, where things do change. Consider the equation below where V is a pre-Hilbert space.

$$Ax = b,$$

  where $A \in L(V, V) \equiv L(V)$ and $x, b \in V$. For $\dim V < \infty$, the commonly known situation is where A is invertible $\iff$ $\det A \neq 0$, and the set of such matrices with matrix multiplication as a group operation is denoted $GL_n(\mathbb{K})$ or sometimes $GL(n, \mathbb{K})$.

  An example of an infinite-dimensional version might be $A \in L(C^\infty(\mathbb{R}), C^\infty(\mathbb{R})) \equiv L(C^\infty(\mathbb{R}))$ with $x, b \in C^\infty(\mathbb{R})$. One such A "could" be of the operator associated with **Lemma 1** on the prior page of this document, $d^k/dx^k + \beta$, if y and f happen to be continuous and infinitely differentiable.

Example 4 (*Cauchy-Riemann Equations, part 1)*)**.** Consider a (non-singular) vector field $X = (u(x,y), v(x,y), 0)$ over a simply connected, open subset of $\mathbb{R}^3$ with $\nabla \cdot X = 0$, $\nabla \times X = 0_{\mathbb{R}^3}$. Then there exists another vector field $A = (0, 0, \psi(x,y))$ such that $X = (\psi_y, -\psi_x, 0)$, the curl of A. We are also guaranteed existence of a smooth, scalar function φ such that $X = (\varphi_x, \varphi_y, 0)$ is its gradient. Then the components satisfy Cauchy-Riemann conditions,

$$\psi_y = \varphi_x \quad \text{and} \quad -\psi_x = \varphi_y. \tag{11}$$

- ***Exercise:*** Show $4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}$ is the standard Laplace-Beltrami operator $\Delta = \nabla \cdot \nabla$ acting on $C^2(\mathbb{R}^2)$ in Cartesian coordinates.

- Now, we convert the Laplace-Beltrami operator to cylindrical coordinates. We identify transformations $x = \rho\cos\theta$ and $y = \rho\sin\theta$ and inverse transformations $\rho = \sqrt{x^2+y^2}$, $\cos\theta = x/\sqrt{x^2+y^2}$, and $\sin\theta = y/\sqrt{x^2+y^2}$. Then we proceed with the chain rule for differential operators

$$\frac{\partial}{\partial x} = \frac{\partial \rho}{\partial x}\frac{\partial}{\partial \rho} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} \quad \text{and} \quad \frac{\partial}{\partial y} = \frac{\partial \rho}{\partial y}\frac{\partial}{\partial \rho} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta}. \tag{12}$$

After making the proper substitutions, we find the derivatives of ρ,

$$\frac{\partial \rho}{\partial x} = x/\rho = \cos\theta \quad \text{and} \quad \frac{\partial \rho}{\partial y} = y/\rho = \sin\theta, \tag{13}$$

and those of θ,

$$\frac{\partial \theta}{\partial x} = -\sin\theta/\rho \quad \text{and} \quad \frac{\partial \theta}{\partial y} = \cos\theta/\rho. \tag{14}$$

Next, we need the second derivatives

$$\begin{aligned} \frac{\partial^2}{\partial x^2} &= \left(\cos\theta\frac{\partial}{\partial \rho} - \frac{\sin\theta}{\rho}\frac{\partial}{\partial \theta}\right)\left(\cos\theta\frac{\partial}{\partial \rho} - \frac{\sin\theta}{\rho}\frac{\partial}{\partial \theta}\right) \\ \text{and} \quad \frac{\partial^2}{\partial y^2} &= \left(\sin\theta\frac{\partial}{\partial \rho} + \frac{\cos\theta}{\rho}\frac{\partial}{\partial \theta}\right)\left(\sin\theta\frac{\partial}{\partial \rho} + \frac{\cos\theta}{\rho}\frac{\partial}{\partial \theta}\right). \end{aligned} \tag{15}$$

Putting it all together we have $\Delta = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$.

Remark. It is likely that most may not want to go through that conversion tedium we just did. This was precisely the point and may add some motivation as to why the equation

$$w^2 f''(w) + wf'(w) + [w^2 - n^2]f(w) = 0, \qquad w \in \mathbb{C}, \tag{16}$$

solved by Bessel function $J_{\pm n}(w)$ solutions, allows the functions $F_{n,k}(\rho,\theta,z) := J_n(k\rho)e^{in\theta}e^{kz}$ to graciously solve the Laplace equation in 3D cylindrical coordinates; this is **Theorem 2.19** (page 17).

- ***Exercise:*** In **Remark 2.8** (*Cauchy-Riemann Equations*, page 11) and just before, the notes discuss the equivalence of differentiability conditions of a map $f : \mathbb{C} \to \mathbb{C}$ and how its real and imaginary parts, say $u, v : \mathbb{R}^2 \to \mathbb{R}$ for $f = u + iv$, must satisfy a Cauchy-Riemann (CR) condition.

Prove it explicitly by showing

$$\frac{\partial f}{\partial \bar{z}} = 0 \iff \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} . \tag{17}$$

- **Theorem 2.6** (page 12, word-for-word reproduced here) If $f : \Omega \to \mathbb{C}$ is a continuous function, then for any continuous curve $\Gamma \subset \Omega$, the following integrals exist:

$$I_1 = \int_\Gamma f(z)\, dz, \qquad I_2 = \int_\Gamma f(z)\, d\ell, \qquad I_3 = \int_\Gamma |f(z)|\, d\ell, \tag{18}$$

where $d\ell = |dz|$ is the arclength element on Γ. Also, $|I_1| \le I_3, |I_2| \le I_3$.

Proof. Fix a partition Δz_k of Γ, i.e. $\Delta z_k = z_k - z_{k-I}$ with $k = I, \cdots, n$, and let c_k be any points on Γ such that c_k lies on the arc from z_{k-I} to z_k. Assume f is integrable along Γ. Then we have

$$\begin{aligned} &\int_\Gamma f(z)\, dz = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k) \cdot \Delta z_k = I_1 \\ \text{and} \quad &\int_\Gamma f(z)\, d\ell = \int_\Gamma f(z)\, |dz| = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k) \cdot |\Delta z_k| = I_2, \end{aligned} \tag{19}$$

which gives

$$\begin{aligned} |I_1| = \left| \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k) \cdot \Delta z_k \right| &= \lim_{n\to\infty} \left| \sum_{k=1}^{n} f(c_k) \cdot \Delta z_k \right| \\ &\le \lim_{n\to\infty} \sum_{k=1}^{n} |f(c_k)| \cdot |\Delta z_k| \qquad (\because \textit{triangle inequality}) \\ &= \int_\Gamma |f(z)|\, |dz| = \int_\Gamma |f(z)|\, d\ell \end{aligned} \tag{20}$$

and

$$\begin{aligned} |I_2| = \left| \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k) \cdot |\Delta z_k| \right| &= \lim_{n\to\infty} \left| \sum_{k=1}^{n} f(c_k) \cdot |\Delta z_k| \right| \\ &\le \lim_{n\to\infty} \sum_{k=1}^{n} |f(c_k)| \cdot |\Delta z_k| \\ &= \int_\Gamma |f(z)|\, d\ell. \end{aligned} \tag{21}$$

□

- *For geometry fans*: We also discussed how the following theorem looks similar to Green's Theorem (one of many cases of the generalized Stokes-Cartan Theorems). Here it is: **Theorem 2.8** (Vekua, page 12, word-for-word reproduced here) Let Ω be a domain in $\mathbb{C}$ with continuous boundary $\partial\Omega$, and Γ a closed curve such that the domain bounded by Γ, $\text{Int}(\Gamma)$, is a subset of Ω. Then

$$\oint_\Gamma f(z) dz = 2i \iint_{\text{Int}(\Gamma)} \frac{\partial f}{\partial \bar{z}} dx dy \tag{22}$$

for any function $f = u + iv$ with $u(x,y), v(x,y)$ differentiable in Ω and continuous on $\partial\Omega$. For reference, Green's theorem may be expressed as follows. Consider two smooth functions $u, v : \mathbb{R}^2 \to \mathbb{R}$ and let $\Omega \subset \mathbb{R}^2$ be bounded by a positively oriented, piecewise smooth, simple closed curve $\partial\Omega \equiv \Gamma$ (simple as in non-intersecting, closed as in $\Gamma(a) = \Gamma(b)$ where the curve $\Gamma : [a,b] \to \Omega$; they are also referred to as Jordan curves). Then, **Green's Theorem** (a special case of Stokes-Cartan) states

$$\oint_{\Gamma=\partial\Omega} (u\, dx + v\, dy) = \iint_{\Omega} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\, dxdy. \tag{23}$$

Lemma 2. *Vekua's Theorem is a less restrictive case of the generalized Stokes-Cartan Theorem,*

$$\int_{\partial\Omega} \omega = \int_{\Omega} d\omega, \tag{24}$$

in the sense that the boundary need only be continuous and not smooth. Here, ω is a differential 1-form and d the exterior derivative.

Proof. Assuming boundary conditions of Vekua's Theorem, fix the complex, differential 1-form $\omega \equiv f\, dz \Rightarrow d\omega = \frac{\partial f}{\partial \bar{z}} d\bar{z}dz = \frac{\partial f}{\partial \bar{z}}(dx - idy) \wedge (dx + idy) = \frac{\partial f}{\partial \bar{z}}(idxdy - idydx) = 2i\frac{\partial f}{\partial \bar{z}} dxdy.$ □

Example 5. Let f, g be smooth, complex-valued functions differentiable in Ω and continuous on $\partial\Omega$. Then consider the complex, differential 1-form $\omega := f\, dz + g\, d\bar{z} \Rightarrow d\omega = \left(\frac{\partial f}{\partial \bar{z}} - \frac{\partial g}{\partial z}\right) d\bar{z}dz$. To write another Stokes-like equation, we can either write it in complex form as

$$\int_{\partial\Omega} (f\, dz + g\, d\bar{z}) = \left(\frac{\partial f}{\partial \bar{z}} - \frac{\partial g}{\partial z}\right) d\bar{z}dz \tag{25}$$

or find the associated, real differential 1- and 2-forms. We just found the associated real form of $d\bar{z}dz$. For ω, we have

$$\begin{aligned} \int_{\partial\Omega} (g\, dz + h\, d\bar{z}) &= \int_{\partial\Omega} [g(dx + idy) + h(dx - idy)] \\ &= \int_{\partial\Omega} (g + h)dx + i(g - h)dy. \end{aligned} \tag{26}$$

Thus another generalization of Stokes-Cartan is

$$\int_{\partial\Omega} [(g + h)dx + i(g - h)dy] = 2i \int_{\Omega} \left(\frac{\partial f}{\partial \bar{z}} - \frac{\partial g}{\partial z}\right) dxdy. \tag{27}$$

- ***Exercise*** *(for geometry fans):* Carry on Stokes-Cartan generalizations for complex differential forms by starting with a 0-form ω and, separately, by starting with a 2-form ω.

- We return to the $Ax = b$ problem along with **Lemma 1**. In this case, take $A \mapsto \mathcal{D} = P_n(d/dx)$ to be a polynomial differential operator acting on functions $f : \mathbb{R} \to \mathbb{R}$ and put $g \in L^1(\mathbb{R})$ (equivalence classes of integrable functions on the line) then the inhomogeneous differential equation $\mathcal{D}f = g$, by **Theorem 2.13** has solution

$$f = \int_{\mathbb{R}} \mathcal{G}_{\mathcal{D}}(x, x')g(x')dx', \tag{28}$$

with

$$\mathcal{G}(x, x') = \frac{1}{2\pi} \int_{\mathbb{R}} \exp[ik(x - x')]/P_n(ik)dk.$$

Example 6. Similar to **Example 2.9** and **Example 2.10** in the notes (page 15), take the system of differential equations from **Lemma 1** with polynomial differential operator $\mathcal{D} = \frac{d^n}{dx^n} + \beta$, $\beta \in \mathbb{R}$. The Green's function for $\mathcal{D}$ is then given by

$$\mathcal{G}(x, x') = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{ik(x-x')}}{1 - k^n} dk \tag{29}$$

Exercise: Find the inhomogeneous solution for the equation $f^{(k)} + \beta f = e^{-|x|}$ after computing its Green's function.

Remark. In the same spirit as the challenge after **Lemma 1**, consider inductively adding terms of degree $< k$ to the polynomial differential operator in the last exercise (equivalent to permuting 1's in the matrix of the lemma) and repeat the exercise for the new operator.